Inverting the Laplace transform

Ex We found

$$Y(s) = \frac{s-1}{s^2-s-2}$$

so let's go back to the t-domain.

We know

- $\mathcal{L}\{c\} = \frac{c}{s}$, c is a constant
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$
- $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

Rewriting $Y(s)$ using partial fractions, we have

$$Y(s) = \frac{s-1}{(s-2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1}$$

so

$$A(s+1) + B(s-2) = s-1$$

$$s(A+B) + A - 2B = s - 1$$

$$A+B=1 \qquad A-2B = -1$$

$$A = 1-B \Rightarrow 1-B-2B = -1$$

$$-3B = -2$$

$$B = \frac{2}{3}$$

$$\Rightarrow A = \frac{1}{3}$$

Therefore,

$$Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

Now, $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{s+1}\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Thus,

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \quad .$$

Chapter 7: Systems of First-Order Linear Equations

§7.1 Introduction

We will consider systems of ODEs and denote the independent variable by t, and the dependent variables as $x_1, x_2, \ldots, x_n$ which are functions of t.

Derivatives will be denoted as $\frac{dx_1}{dt}$ or x_1'
($\dot{x}_1$ is also common)

Why study systems of ODEs?

Higher order equations can be reformulated as a system of first order ODEs.

Ex Consider the general equation for a spring-mass system

$$mu'' + \gamma u' + ku = F(t).$$

Let $x_1 = u$ and $x_2 = u'$. Then

$$x_1' = u' = x_2$$

$$x_2' = u'' = \frac{1}{m}(F(t) - \gamma u' - ku) = \frac{1}{m}(F(t) - \gamma x_2 - kx_1)$$

and so in matrix form, we have the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F(t)}{m} \end{bmatrix}.$$

For an arbitrary nth order equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

we let

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ &\vdots \end{aligned}$$

$$x_n = y^{(n-1)} \qquad x_n' = y^{(n)} = F(t, x_1, x_2, \dots x_n)$$

More generally, the system of n first order ODEs is

$$x_1' = F_1(t, x_1, x_2, \dots, x_n)$$
$$x_2' = F_2(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$x_n' = F_n(t, x_1, x_2, \dots, x_n)$$

and the solution of such a system consists of n functions

$$x_1 = \varphi_1(t)$$
$$x_2 = \varphi_2(t)$$
$$\vdots$$
$$x_n = \varphi_n(t)$$

We may also prescribe initial conditions

$$x_1(t_0) = x_1^0, \; x_2(t_0) = x_2^0, \; \dots, \; x_n(t_0) = x_n^0 .$$

We can think of the solution as a set of parametric equations in n-dimensional space.

<u>Thm</u> (Existence and Uniqueness)

Let each of the n functions $F_1, \ldots, F_n$ and the n^2 partial derivatives $\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n}$ be continuous in a region R of $t x_1 x_2 \cdots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_n < \beta_n$ and let $(t_0, x_1^0, x_2^0, \ldots, x_n^0) \in R$. Then there is an interval $|t_0 - t| < h$ in which there exists a unique solution $\vec{x} = \vec{\varphi}$ of the system of ODEs that satisfies the IC.

Linear and Nonlinear Systems

If $F_1, \dots, F_n$ are linear functions of $x_1, \dots, x_n$ then the system is called linear.

Otherwise, it is nonlinear.

General system of n first-order linear ODEs

$$
\begin{aligned}
x_1' &= p_{11}(t)\, x_1 + \cdots + p_{1n}(t)\, x_n + g_1(t) \\
x_2' &= p_{21}(t)\, x_1 + \cdots + p_{2n}(t)\, x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)\, x_1 + \cdots + p_{nn}(t)\, x_n + g_n(t)
\end{aligned}
$$

If $g_i(t) = 0$ for $t \in I$, $i = 1, \dots, n$, the system is called homogeneous. Otherwise it is called nonhomogeneous.

Thm (Existence and Uniqueness for Linear Systems)

If $p_{ij}(t)$ and $g_i(t)$, $i,j = 1, \ldots, n$ are continuous on $I : \alpha < t < \beta$, then there exists a unique solution $\vec{x} = \vec{\varphi}$ that satisfies the ICs. The solution exists throughout the interval I.

We will spend the remainder of our time studying linear systems of first-order ODEs.

7.2 Matrices

The $m \times n$ matrix with entries a_{ij} is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The transpose of A is denoted A^T and if $A = (a_{ij})$, $A^T = (a_{ji})$

The conjugate of A is denoted $\bar{A}$ and $\bar{A} = (\bar{a}_{ij})$

The adjoint of A is $\bar{A}^T$ or $A^* = (\bar{a}_{ji})$

We will focus on square matrices ($m = n$) and (column) vectors.

Properties of Matrices

1. Equality : A and B are equal if $a_{ij} = b_{ij}$ for each i and j

2. Zero : $\vec{0}$ will denote the matrix or vector whose entries are all zero

3. Addition $A + B = (a_{ij}) + (b_{ij})$

4. Scalar Multiplication

$$\alpha A = \alpha (a_{ij}) = (\alpha a_{ij})$$

5. Matrix Multiplication

$$C = AB$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note Matrix multiplication is not commutative

$$AB \neq BA$$

6. Multiplication of vectors (dot product)

$$\vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i$$

This produces a scalar

Inner product $(\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i \overline{y_i}$

Length/magnitude of $\vec{x}$ is $\|\vec{x}\| = \sqrt{(\vec{x}, \vec{x})}$

- if $(\vec{x}, \vec{x}) = 0$, must have that $\vec{x} = \vec{0}$.

If $(\vec{x}, \vec{y}) = 0$, then $\vec{x}$ and $\vec{y}$ are orthogonal

7. Identity matrix $\quad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

$$AI = IA = A$$

8. Inverse and Determinant

A is called nonsingular/invertible if there is a matrix B so that $AB = I$ and $BA = I$. We write $B = A^{-1}$.

If A has no inverse, we call A singular or noninvertible.

A is nonsingular $\Longleftrightarrow \det A \neq 0$.

Matrix Functions

We will consider vectors and matrices whose elements are functions.

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

$A(t)$ is called continuous if each element of A is continuous.

Similarly, $$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt} \right)$$

$$\int_a^b A(t)\,dt = \left(\int_a^b a_{ij}\,dt\right)$$

Ex $$A(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix}$$

$$A'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}$$

$$\int_0^{\pi} A(t)\,dt = \left.\begin{pmatrix} -\cos t & \frac{t^2}{2} \\ t & \sin t \end{pmatrix}\right|_0^{\pi}$$

$$= \begin{pmatrix} 2 & \frac{\pi^2}{2} \\ \pi & 0 \end{pmatrix}$$

Many calculus rules hold:

$$\frac{d}{dt}(CA) = C\frac{dA}{dt}$$ where C is a constant matrix

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$$

7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues and Eigenvectors

Systems of Linear Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

or

$$A\vec{x} = \vec{b}$$

If $\vec{b} = 0$, the system is called homogeneous. Otherwise it is nonhomogeneous.

If A is nonsingular, then $\vec{x} = A^{-1}\vec{b}$, and the homogeneous problem only has the solution $\vec{x} = \vec{0}$.

If A is singular, then $A\vec{x} = \vec{b}$ has no solution or a nonunique solution. The homogeneous problem $A\vec{x} = \vec{0}$ has infinitely many solutions.

When does $A\vec{x} = \vec{b}$ have a solution?

We must have that $(\vec{b}, \vec{y}) = 0$ for all $\vec{y}$ satisfying $A^*\vec{y} = \vec{0}$.

Then there are infinitely many solutions of the form

$$\vec{x} = \vec{x}^{(0)} + \vec{\xi}$$

where

$$A\vec{x}^{(0)} = \vec{b} \quad \text{and} \quad A\vec{\xi} = \vec{0} \, .$$

Linear Dependence and Independence

$\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ are linearly dependent if there are scalars $c_1, \ldots, c_k$ (with at least one nonzero) such that

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + \cdots + c_k \vec{x}^{(k)} = \vec{0}$$

If the only solution is $c_1 = c_2 = \cdots = c_k = 0$, then $\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}$ are linearly independent.

Eigenvalues and Eigenvectors

We may consider $A\vec{x}=\vec{y}$ as a linear transformation that maps $\vec{x}$ to $\vec{y}$. We are interested in finding vectors that are mapped to a scalar multiple of itself. I.e. $\vec{y}=\lambda\vec{x}$. Then

$$A\vec{x}=\lambda\vec{x}$$

$$A\vec{x}-\lambda\vec{x}=0$$

$$(A-\lambda I)\vec{x}=0$$

This equation has nonzero solutions if and only if

$$\det(A-\lambda I)=0 \quad \text{(characteristic equation)}$$

λ is called an eigenvalue

$\vec{x}$ is called an eigenvector

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